Maitra Cascade Minimization - PRELIMINARY VERSION

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Abstract

In this paper an efficient algorithm for the synthesis and minimization of CA *(cellular array architectures)* is proposed. Our algorithm starts from a completely-specified switching function and produces reversible wave cascades with an effort to minimize the number of the produced chains (cascades). This kind of topology can easily be mapped to reversible circuits (generalized Toffoli gates) as it is presented in the related bibliography. The proposed algorithm uses function decomposition and ETDDs (EXOR ternary decision diagrams). The experimental results that are obtained (they are presented at the end of this work) prove the efficiency of the proposed method.

1 Introduction

Cellular architectures (CA) are characterized by several attractive properties. They have relatively small logic blocks with local (and sometimes limited) interconnection between them. Moreover, with the creation and evolution of LUT-based FPGAs and CA-type FPGAs [8] these architectures can easily be implemented.

One of the simplest forms of cellular architectures are the reversible wave cascades (or Maitra cellular arrays), because they require very simple cells (Maitra cells) with limited interconnection between them (Fig. 1) and every cell implements a two-variable switching function. Many cells are linked as a chain (Maitra cascade [1] or complex term) and all the chains are linked together through an XOR-collector row.

In Ref [4] it was shown that a reversible wave cascade could be directly mapped to reversible logic gates and more specifically to *Generalized Toffoli gates*. A logic gate is called reversible if it has the same number of inputs, outputs, and maps each input vector into a unique output vector and vice versa. One of their important properties is that they consume minimal amounts of power due to the fact that they lose no information[16]. In Ref [12] it was shown that all quantum logic gates must also be reversible. Due to this, the reversible wave cascades is a very attractive architecture for the implementation of reversible logic circuits and even quantum logic.

There were algorithms developed in the past, for mapping switching functions to CA architectures, although they are still immature. Moreover, most of those algorithms create topologies rather different than the one presented in this paper (with notable exception Ref [4]).

In references [5, 6, 7, 8, 9] techniques like variable reordering and cube transformation or even multi-value logic are used in an effort to minimize the number of complex terms, though their architectures differ from the one presented in this paper and are more complex. In references [14, 15] a systematic method was presented to produce architectures very similar to Maitra cellular array architectures, by extending the EXORLINK operation to complex terms. In Ref [4] an algorithm was proposed for mapping Maitra cellular arrays to reversible gates (more specifically Generalized Toffoli gates), although it was not implemented. Moreover, none of the above algorithms guarantees minimality. In Ref [17] two algorithms were presented for minimizing the number of cascades in a reversible wave cascade. The first one guaranteed minimality for functions up to 5 variables.The second algorithm applied the first one on groups of cascades, inside the cellular array, as a complex term transformation operation. This procedure was repeated several times, over different groups of cascades, in an effort to minimize their number.

In this paper we introduce an algorithm that can produce minimal expressions (the ones with the least number of complex terms) for a switching function f with at most 5 variables. This algorithm has been the basis for the creation of a heuristic one that produces near optimal results for functions with more than 5 variables. The method proposed here gives better results than the ones in Ref [8, 14] and is much more efficient than the one presented in Ref [17].

2 Theoretical background

In this section we provide some background definitions. An expression of a switching function suitable for mapping to a Maitra cascade(cell chain) is called a Maitra term. A more formal definition[4] follows:

Definition 1. A complex Maitra term (complex term or Maitra term for simplicity) is recursively defined as follows:

1. Constant 0 (1) Boolean function is a Maitra term.
2. A literal is a Maitra term.
3. If M_i is a Maitra term, a is a literal, and G is an arbitrary two-variable Boolean function, then $M_{i+1} = G(a, M_i)$ is a Maitra term.

Additionally, it is required that each variable appears in each Maitra term only once.

A complex term can be directly mapped to a reversible gate and more specifically to a generalized Toffoli gate. A $k * k$ generalized Toffoli gate is defined as: $P_1 = A_1, P_2 = A_2, \dots, P_{n-1} = A_{n-1}, P_n = f_{n-1}(A_1, A_2, \dots, A_{n-1}) \oplus A_n$, where A_i are the inputs of the gate, P_i are the outputs of the gate and f_{n-1} is an arbitrary switching function of $n-1$ variables [4].

Definition 2. A reversible wave cascade expression (or Maitra expression) for a switching function is an exlusive-OR sum of complex terms:

$$Q = \sum_{i=1}^{m} \oplus M_i,$$

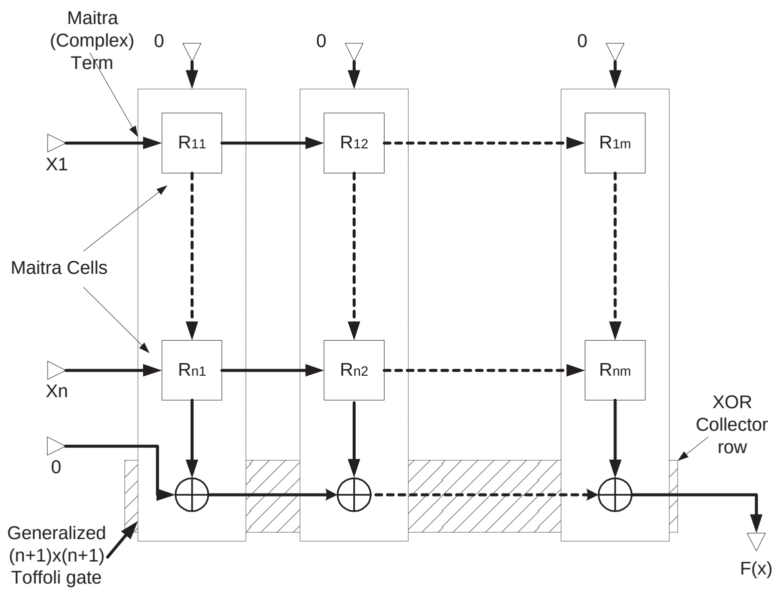

Figure 1: Reversible wave cascade CA

where m is the number of complex terms. All complex terms M_i inside Q have the same variable ordering.

A reversible wave cascade expression is considered a reversible gate because it consists of reversible gates (Fig. 1).

Definition 3. A minimal (or exact) expression of a switching function $f(x_1, \ldots, x_n)$ of n variables, is defined as the wave cascade expression which has the fewest number of complex terms comparing to every other wave cascade expression for this function.

Definition 4. The weight $w(f)$ of a switching function $f(x_1, \ldots, x_n)$ of n variables is defined as the number of complex terms in a minimal expression of f.

Definition 5. A switching function is called cascade realizable, if it has weight 1.

Every two-variable switching function is cascade realizable although not every such function can be presented as one product term (i.e. a complex term where only logical AND is allowed between literals). More specifically, functions: $f = \bar{x_1}x_2 \oplus x_1\bar{x_2}, f = \bar{x_1}x_2 \oplus \bar{x_2}, f = \bar{x_1}\bar{x_2} \oplus x_1x_2, f = \bar{x_1}\bar{x_2} \oplus x_2, f = \bar{x_1}\bar{x_2} \oplus x_1, f = \bar{x_1}x_2 \oplus x_1$ are cascade realizable, but they consist of at least two product terms. The remaining two-variable switching functions can be implemented as one complex term and as one product term.

A wave cascade expression can be directly mapped to a reversible wave cascade cellular architecture(Fig. 1).

It has been proved[2][3] that a Maitra cell doesn't need to implement every two-variable switching function. A set of only six functions is sufficient (complete set). Of course there are many equivalent such sets[13]. We have adopted

Table 1: Cell index set

Cell index(r)	$F_r(x,y)$
1	$x+y$
2	$\overline{x}+y$
3	$\overline{x}y$
4	xy
5	$x \oplus y$
6	y

(in the rest of the paper) one of them which can be seen in Table 1. The cascades that use cells which implement any switching function from the complete set are called *Restricted Maitra Cascades* [3] and lead to smaller implementations, since only three bits per Maitra cell are required instead of four. From this point on, without loss of generality, when we mention Maitra cascades, we will refer to restricted Maitra cascades.

It is assumed that the first cell in every cascade has one of its inputs connected to 0, which means that it has index 1,2 or 6.

2.1 Representation

A complex term is characterized by its cells, since its first input is the constant 0. Therefore, we can represent it by a series of cells (using the corresponding index shown in table 1), utilizing three bits per each. The leftmost cell has one of its inputs hardwired to constant 0 (represents the first cell of the cascade) and the rightmost is the cell which connects to the XOR collector.

For example function $f(x_1,x_2,x_3,x_4) = (x_1 \oplus x_2)x_3 + x_4$ is cascade realizable. Using cells from the previously defined cell set, it can be represented as: 1541 or in bits: 001101100001. Likewise function $f(x_1,x_2,x_3,x_4) = \{(x_1 \oplus x_2)x_3 + x_4\} \oplus \{x_1 + x_2 + x_3 + x_4\}$ can also be represented as: 1541 $\oplus$ 1111.

2.2 Decompositions

Every boolean function can be expressed with the help of its subfunctions through relations known as boolean decompositions (or expansions).

Definition 6. Let $f(X)$ be a switching function and X the vector of its variables. Let x_1 be one of the variables in the vector X. Then, $f(x_1 = 0, x_2, \ldots)$, $f(x_1 = 1, x_2, \ldots)$ and $\{f(x_1 = 0, x_2, \ldots) \oplus f(x_1 = 1, x_2, \ldots)\}$ are subfunctions of f, regarding variable x_1. For simplicity, in the rest of this paper, we will refer to $f(x_1 = 1, x_2, \ldots)$ as f_1, to $f(x_1 = 0, x_2, \ldots)$ as f_0, to $\{f(x_1 = 0, x_2, \ldots) \oplus f(x_1 = 1, x_2, \ldots)\}$ as f_2 and to x_1 as x.

A boolean function f can be expressed as:

$f(X) = \bar{x}f_0 \oplus xf_1$, $f(X) = xf_2 \oplus f_0$, $f(X) = \bar{x}f_2 \oplus f_1$ (Shannon, positive Davio, negative Davio).

Definition 7. Let f be an n-variable switching function. By applying the Shannon and Davio decompositions on f, a ternary tree (ETDD-Exor Ternary

Decision Diagram) is generated. This decomposition is applied until the constant 0 or 1 function is encountered or a leaf is obtained. This tree is named the generator tree.

It was proved in Ref [17] that a switching function f can be decomposed using expansions, different than Shannon and Davio. Those are presented in Theorem 1.

Theorem 1 (New Decompositions)**.** *Given a Boolean function $f(X)$, where X is the vector of the function's variables, and a variable x of this vector, we can express f as:*

$$f(X) = (x + f_2) \oplus (x \oplus f_1) \tag{1}$$

$$f(X) = (x + f_0) \oplus (x\bar{f}_1) \tag{2}$$

$$f(X) = (x\bar{f}_2) \oplus (x \oplus f_0) \tag{3}$$

$$f(X) = (x + \bar{f}_0) \oplus (\bar{x} + \bar{f}_1) \tag{4}$$

$$f(X) = (\bar{x} + \bar{f}_2) \oplus \bar{f}_0 \tag{5}$$

$$f(X) = (x + \bar{f}_2) \oplus \bar{f}_1 \tag{6}$$

$$f(X) = (\bar{x}\bar{f}_2) \oplus (x \oplus \bar{f}_1) \tag{7}$$

$$f(X) = (\bar{x}\bar{f}_0) \oplus (\bar{x} + f_1) \tag{8}$$

$$f(X) = (\bar{x} + f_2) \oplus (x \oplus \bar{f}_0) \tag{9}$$

For the proof refer to [17].

The classic Shannon and Davio expansions create expressions composed by product terms. The expansions presented in Theorem 1 create expressions composed by complex terms [17], so they cannot be used in ESOP minimization.

Lemma 1. *The relation*
$G_{r_1}(x, y_1) \oplus G_{r_2}(x, y_2) = G_r(x, y_1 \oplus y_2)$ *(G is an ESCT cell function), x, y_1, y_2 are binary variables and $y_1 \neq y_2$ and $y_1 \neq \bar{y_2}$ is true iff:*
$(r_1, r_2, r) = (1, 1, 3), (1, 3, 1), (2, 2, 4), (2, 4, 2), (3, 3, 3),$
$(4, 4, 4), (5, 5, 6), (5, 6, 5), (6, 6, 6)$
Proof. The above lemma can easily be proved exhaustively [11]. Q.E.D.

If $y_1 = y_2$ or $y_1 = \overline{y_2}$ then $y_1 \oplus y_2$ and consequently $F_r(x, y_1 \oplus y_2)$ are reduced to one complex term[17].

For example, two complex terms: $(x_1 + \overline{x_2})\overline{x_3}x_4$ (or in the form presented in section 2.1: 1234) and $((x_1 + \overline{x_2}) \oplus x_3) + \overline{x_4}$ (or 1252) can be merged (in respect to variable x_4) as follows: $[(x_1 + \overline{x_2})\overline{x_3}x_4] \oplus [((x_1 + \overline{x_2}) \oplus x_3) + \overline{x_4}] = \{[(x_1 + \overline{x_2})\overline{x_3}] \oplus [((x_1 + \overline{x_2}) \oplus x_3)]\} + \overline{x_4}$ or $1234 \oplus 1252 = (123 \oplus 125)2$.

The above lemma shows that we can merge any number of index 1 or 3 cells to one cell of index 1 or 3 and the same principle applies to cells of index 2,4 and 5,6. Therefore, the lemma implies that there are three different sets of cells (cell classes). Any number of cells belonging to the same class is reduced to one cell of the same class. The first class is composed of cells with index 1,3, the second of cells with index 2,4 and the last one of cells with index 5,6. For the cells that one of their inputs is constant 0, we have two cell classes, the first one is composed of cells with index 1,2 and the second of cells with index 6.

Table 2: XOR-sum of a complex term with X or $\overline{X}$

p	q	y_1	r	y_2
1	3	y	1	$\bar{y}$
3	1	y	3	$\bar{y}$
2	2	$\bar{y}$	4	y
4	4	$\bar{y}$	2	y
5	6	y	6	$\bar{y}$
6	5	y	5	$\bar{y}$

Theorem 2 (Complement complex term)**.** *The complement function of a complex term is also a complex term. In the complement complex term, all cells belonging to class(1,3) (i.e. functions $+x, \cdot\bar{x}$) or (2,4) (i.e. functions $+\bar{x}, \cdot x$) are replaced by the others of the same class. Cells of index 5 (i.e. function $\oplus x$) or 6 (i.e. function $\cdot 1$) remain the same in the complement complex term. For the cells of the complex term that have one of their inputs hardwired to constant 0, function $+x$ (cell of index 1) changes to $+\bar{x}$ (cell of index 2) and vice versa.*

Proof. It can easily be proved exhaustively, using induction (start from a simple cell). Q.E.D.

Corollary 1. *A switching function and its complement have the same weight. Q.E.D.*

Theorem 3 (Complex term $\oplus x_n$)**.** *The result of the XOR-sum of a complex term $f = F_n(x_n, F_{n-1}(x_{n-1}, \ldots F_1(x_1, 0))$, where F_i are Maitra cells, with x_n (the variable corresponding to the last cell of the complex term) is also a complex term.*

Proof. It can easily be proved exhaustively. Q.E.D.

Corollary 2 (Complex term $\oplus\overline{x_n}$)**.** *The result of the XOR-sum of a complex term $f = F_n(x_n, F_{n-1}(x_{n-1}, \ldots F_1(x_1, 0))$, where F_i are Maitra cells, with $\overline{x_n}$ (the variable corresponding to the last cell of the complex term) is also a complex term. Q.E.D.*

The rules to create such expressions are presented in Table 2. The starting complex term is: $F_p(x_n, y)$. Complex terms F_p, F_q are: $F_q(x_n, y_1) = F_p(x_n, y) \oplus x_n$ and $F_r(x_n, y_2) = F_p(x_n, y) \oplus \overline{x_n}$.

For example let $f = (x_1 \oplus x_2)x_3 + \overline{x_4}$ be a complex term or in the representation presented in section 2.1 (we will be using this representation for the rest of this paper): 1542. The complement complex term of f is: $\bar{f} = 2524$. The XOR-sums of f with x_4 and $\overline{x_4}$ are: $f \oplus x_4 = 2522$ and $f \oplus \overline{x_4} = 1544$.

It is important to note that the application of theorems 2, 3 and corollary 2 to a complex term P does not alter the cell class of its cells.

2.3 Minimization theorems

The following theorems present the concept of normalized form and prove that a minimal expression of a function f can be found from the minimal expressions of its subfunctions, as long as, the number of variables of f is less than 6.

Definition 8. An equivalent expression (F_2) of a reversible wave cascade expression (F_1) for a switching function $f(x_1, \ldots x_n)$ is an expression produced by applying Theorem 2, 3 or Corollary 2 to pairs of complex terms in the expression F_1 or by applying the above operations to pairs of complex terms inside expressions of subfunctions in the generator tree of f.

For example if $F_1 = 1234 \oplus 2343$, then an equivalent expression of F_1 by applying Theorem 3 is: $F_2 = 1234 \oplus x \oplus 2343 \oplus x = 2414 \oplus 2341$.

Theorem 4. *Each minimal expression of a switching function f can always be written in one of the following normalized forms (composed of normalized complex terms F_p, F_q, F_r):*

$$f = F_p(x_1, y) \tag{10}$$

with $(p, y) = (1, f_0)$ and $f_1 = 1$, $(2, f_1)$ and $f_0 = 1$, $(3, f_0)$ and $f_1 = 0$, $(4, f_1)$ and $f_0 = 0$, $(5, f_0)$ and $f_2 = 1$, $(6, f_0)$ and $f_2 = 0$

OR

$$f = F_p(x_1, y) \oplus F_q(x_1, z) \tag{11}$$

with $(p, q, y, z) = (3, 4, f_0, f_1), (3, 6, f_2, f_1), (4, 6, f_2, f_0)$.

OR

$$f = F_p(x_1, y) \oplus F_q(x_1, z) \oplus F_r(x_1, g) \tag{12}$$

with $p = 3, q = 4, r = 6$ and $y \oplus z = f_2$, $y \oplus g = f_0$, $z \oplus g = f_1$.

Every such form has equivalents that can be produced according to the Definition 8.

Proof. Every minimal expression of f (in the form of exclusive-or sum of complex terms) will be:

$$f(x_1 \ldots x_n) = F_{r_1}(x_1, y_1) \oplus F_{r_2}(x_1, y_2) \oplus \ldots \oplus F_{r_n}(x_1, y_n)$$

where F_{r_i} are Maitra cells and $y_1, y_2, \ldots y_n$ are cascade realizable functions. Because of Lemma 1 the above equation can be composed of at most three normalized complex terms, with their last cells belonging to different cell classes.

Function f can also be expressed using the Shannon expansion. By comparing the Shannon expansion with equations 10,11,12, we obtain the forms presented in the theorem, along with their equivalents. Q.E.D.

The equivalent forms presented above, create expressions in the form of XOR-sum of complex terms and, as it can easily be observed, they constitute the expressions produced by the boolean decompositions presented in Theorem 1. So these equivalent forms can be used in place of those expansions.

Theorem 5. *At least one minimal expression of a switching function $f(x_1, \ldots, x_n)$ with less than 6 variables ($n < 6$) can be obtained from the minimal expressions of f_0, f_1, f_2.*

Q.E.D.

For example switching function $f = 128e$ (refer to Fig. **??**) has weight 3. Two minimal expressions of its subfunctions f_1 and f_0 are: $\{(235) \oplus (621)\}$ and $\{(235) \oplus (152)\}$ respectively. These two expressions have one common term, which will be merged when producing expressions for f. The final solution will be: $(2354) \oplus (6214) \oplus (2353) \oplus (1523) = (2356) \oplus (6214) \oplus (1523)$.

It is proved in [10] that a function f with 4 variables has: $w(f) \leq 3$. According to Theorem 5 all minimal expressions of a function f with weight $w(f) \leq 3$ can be found and thus, all minimal expressions for every function with 4 variables can be obtained. So we can find at least one minimal expression of a switching function with 5 variables.

2.4 Generator Terms

Theorem 5 indicates that, in order to produce a minimal expression for a switching function f we must have all minimal expressions of its subfunctions, including their equivalents. As shown in Theorem 4, this can produce many expressions. In the next theorems it is proved that we can find minimal solutions for f without producing any equivalent forms.

Definition 9. The representative cell for cell class (1,3) is 3. The representative cell for cell class (2,4) is 4 and for (5,6) is 6. If the cell has one input hardwired to constant 0, then the representative for cell class (1,2) is 1 and for cell class (6) is 6.

Definition 10. A generator complex term is a complex term composed from representative cells.

Definition 11. Two complex terms have the same generator complex term if their corresponding cells belong to the same cell class. Those two complex terms are called relatives.

Definition 12. Two reversible wave cascade expressions belong to the same generator class if for every complex term (P_1) in the first expression, there is a complex term in the second expression which has common generator complex term with P_1.

For example complex terms 1234 and 1414 have the same generator complex term 1434 (they are relatives), because all their corresponding cells belong to the same cell class. Complex term 1434 is a generator complex term because all its cells are representatives of their cell classes. The following two reversible wave cascade expressions: $Q_1 = 1234 \oplus 6215$ and $Q_2 = 1414 \oplus 6116$ belong to the same generator class since complex terms 1234 and 1414 have the same generator complex term (1434) and complex terms 6215 and 6116 also have the same generator complex term (6136).

Those last definitions create classes of relative complex terms and equivalent reversible wave cascade expressions.

Lemma 2. *If $P_1 = c_{11}c_{12}\ldots c_{1n}$ and $P_2 = c_{21}c_{22}\ldots c_{2n}$ are two complex terms with the same generator complex term (c_{1i}, c_{2i} are Maitra cells of the same class) then $P_1 \oplus P_2$ is also a complex term and moreover if complex terms $c_{11}c_{12}\ldots c_{1i}$ and $c_{21}c_{22}\ldots c_{2i}$, $i < n$ are equals or complements then the cells between i+2 and n of $P_1 \oplus P_2$ will be of the same cell class with those of P_1 and P_2. The rest of the cells will represent literal x_{i+1} or $\overline{x_{i+1}}$ (using the representation in section 2.1, the cells will be : $\underbrace{6\ldots6}_{(i)}1$ and $\underbrace{6\ldots6}_{(i)}2$ respectively).*

Proof. It can easily be proved using Lemma 1. Q.E.D.

For example: $1334 \oplus 2334 = 6234$ (complex terms 1 and 2 are complements) or $123455 \oplus 241466 = 666155$ (complex terms 123 and 241 are complements and according to Table 2 it holds: $1234 \oplus 6661 = 2414$).

Corollary 3. *If $M \oplus P = P_1$ and P, P_1 relative complex terms, then complex term M must be of the form presented in Lemma 2. Q.E.D.*

Definition 13. Constant input level (CIL(c)) of a complex term c is the number of cells in the term that have one of their inputs as constant 0.

In other words CIL is the number of cells with index 6 at the start of the cascade. For example: CIL(66612) = 3.

Lemma 3. *The XOR sum of two relative complex terms has CIL at least 1.*

Proof. It can easily be proved. Q.E.D.

Lemma 4. *If $M \oplus P = P_1$, with P, P_1 relative complex terms, and M_1 is a complex term with the same number of cells with M, $CIL(M_1) > CIL(M)$ and cells of the same cell class with those of M from position $CIL(M_1)+2$ until the last cell, then $M_1 \oplus P = P_2$ and P, P_2 are relative complex terms.*

Proof. It can easily be proved using Lemma 1. Q.E.D.

For example: $(1243) \oplus (6123) = (2243)$ and $(1243) \oplus (6613) = (2443)$ and complex terms $(2243), (2443)$ are relatives.

Lemma 5. *The XOR-sum of two complex terms P_1, P_2 (of n cells) with CIL(P_1)<CIL(P_2) and same representative cells between (MAX(CIL(P_1),CIL(P_2))+2) cell and the nth cell, is a complex term and is relative to P_1.*

Proof. It can easily be proved using Lemma 1. Q.E.D.

For example: $666613 \oplus 661433 = 661413$.

Lemma 6. *A complex term $P_1 = c_1c_2 \ldots c_i \ldots c_n$ (c_i are Maitra cells) can be split at cell c_i (position i) into two complex terms Q_1, K_1. The cells of these terms belong to the same cell class with the corresponding cells of P_1 except only at the ith position (ith cell). The ith cells of complex terms P_1, Q_1, K_1 belong to different cell classes.*

Proof. It can easily be proved exhaustively. Q.E.D.

For example: $1234 = 1264 \oplus 1244$ or $1232 = 2411 \oplus 1235$.

Lemma 7. *Let us assume that there are two complex terms P_1, P_2 which have common generator complex term. If we split $P_1 = P_{11} \oplus P_{12}$ and $P_2 = P_{21} \oplus P_{22}$, at the same cell position, according to Lemma 6, then reversible wave cascade expressions $P_{11} \oplus P_{12}$ and $P_{21} \oplus P_{22}$ belong to the same generator class.*

Proof. According to Lemma 6 complex terms P_{11}, P_{12} and P_1 have only one cell of different class (assume it is the ith cell). Complex Terms P_{21}, P_{22}, P_2 will also differ at the ith cell. It is obvious that all the cells of complex terms $P_{11}, P_{12}, P_{21}, P_{22}$ will belong to the same cell class, except the ith cell. There are only 3 cell classes. Let us assume that the ith cell of P_1, P_2 belongs to the first cell class. The ith cell of P_{11} will belong to the second cell class and the ith cell of P_{12} to the third (the last one). The same holds for P_{21} and P_{22}. Thus, P_{11} and one of P_{21}, P_{22} (let's assume here, without loss of generality, P_{12}), will have the same generator complex term. The remaining two complex terms (here P_{12}, P_{22}) will also have the same generator complex term. Q.E.D.

For example, let us assume that $P_1 = 1233$ and $P_2 = 1411$ are two complex terms which have the same generator complex term (1433). If we split (according to Lemma 6) P_1 and P_2 at the last cell (the rightmost cell) then two reversible wave cascade expressions will be created: $Q_1 = 1234 \oplus 1236$ and $Q_2 = 1414 \oplus 1415$. It is obvious that Q_1 and Q_2 belong to the same generator class.

The next theorem proves that, in order to find a minimal solution for a switching function f with less than 6 variables, we don't need all minimal solutions of its subfunctions. We need only one from every set of equivalent solutions. We can produce a minimal solution for f by merging relative terms. That merging will produce a by-product complex term which will be merged with other complex terms in the final expression of f.

Theorem 6. *Let Q_1 be a minimal expression of switching function f, produced by the minimal expressions F_{i1}, F_{j1} of f_i, f_j respectively (f_i, f_j are subfunctions of f). An, equivalent to Q_1, minimal expression Q_2 of f can be obtained from two other minimal expressions F_{i2}, F_{j2} of f_i, f_j provided that these are equivalents to F_{i1}, F_{j1} respectively.*

3 The Minimization algorithm

Based on the previous theorems and lemmas we give an informal description of an algorithm that minimizes the number of complex terms of a reversible wave cascade expression of a switching function. Those expressions produced are minimal for switching functions $f(n)$ with n < 6 and near minimal for functions with n ≥ 6.

The algorithm (EMin1) receives as input a switching function in minterm formulation (a bitvector where a value of 1 at the ith bit denotes that the function contains the ith minterm) and decomposes it using ETDDs. Every function in the ETDD is decomposed, using the standard Shannon and Davio expansions. During the composition (the reverse procedure of decomposition) of an expression by its subfunctions' forms, complex terms with the same generator complex term are merged and a by-product is produced. Following we have the assimilation phase when the algorithm tries to merge the by-product with other complex terms in the expressions. If it succeeds, then the weight of the function is reduced by 1. The final minimal expression of f will be produced by the minimal expressions of its subfunctions.

4 Conclusions

The algorithm EMin1 concerns single output switching functions. The main contribution of this paper can be summarized as follows:

1. New boolean decomposition formulas are given.

2. An algorithm has been described for producing minimal reversible wave cascades for switching functions up to 5 variables and near minimal for functions with more variables.

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